THE ASYMPTOTIC INTEGRATION OF THE SYSTEM OF EQUATIONS FOR THE LARGE DEFLECTION OF SYMMETRICALLY LOADED SHELLS OF REVOLUTION

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The equations for large deflections of symmetrically loaded shells of revolution contain a natural small parameter ε^2 (the relative thinness). With the aid of asymptotic methods it has been shown that for small ε there is an equilibrium state of the shell for which the shell behaves like a membrane everywhere except for a narrow section near the boundary where an edge effect becomes evident. At the same time a practical method of calculating this solution is developed.

1. Formulation of the problem. Consider the system of nonlinear differential equations for the large deflection of symmetrically loaded shells of revolution [1]

$$Av - \frac{u^2}{2} + \theta u = 0 \qquad \left(u = \frac{dw}{d\rho}, \ A(\ ) \equiv -\rho \frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}(\ )\rho\right) \quad (1.1)$$

$$\varepsilon^2 Au + uv - \theta v + \varphi(\rho) = 0, \quad \varphi(\rho) = \frac{1}{Eh}\int_0^\rho q(t)\, t dt, \quad \varepsilon^2 = \frac{h^2}{12(1-\sigma) r_1^2}$$

where w is the deflection of the middle surface of the shell, Ehv/ρ is the radial force, E is Young's modulus, σ is Poisson's ratio, h is the thickness of the shell, ε^2 describes the relative thinness, r_1 is the radius of the external boundary, $q(\rho)$ is the intensity of the normal loading, and θ is the angle of slope of the shell in the undeformed state; in the case of a spherical shell, for instance, $\theta = \theta_1\rho$, where θ_1 is the curvature.

The boundary conditions, when the shell is partially clamped along the boundary, have the form

$$\frac{dv}{d\rho} - \frac{\sigma}{\rho} v = 0 \qquad \left(0 < \sigma < \frac{1}{2}\right)$$
$$u = 0 \quad \text{when } \rho = 1, \qquad \frac{v}{\rho} < \infty, \quad \frac{u}{\rho} < \infty \quad \text{when } \rho = 0 \tag{1.2}$$

(such a type of boundary clamping has been chosen only for the sake of definiteness; it can easily be changed later to some other common case, such as hinged support).

We will investigate the asymptotic behavior of the solutions of the problem (1.1,2) as $\varepsilon \to 0$. In the case of a plate ($\theta = 0$), the relevant research has been carried out in the work [2], where it was established that the solution of the problem (1.1,2) is close to the solution of the "degenerate" solution (the membrane problem) everywhere except in a small neighborhood of the boundary $\rho = 1$ where there is an edge effect. For this it was essential that, both in the degenerate, as well as in the non-degenerate problem, there is uniqueness of the solution.

In the following, an important role is played by the degenerate problem (on the equilibrium of a membrane)

$$Av_0 - \frac{u_0^2}{2} + \theta u_0 = 0, \qquad u_0 v_0 - \theta v_0 + \varphi(\rho) = 0 \tag{1.3}$$

$$\frac{dv_0}{d\rho} - \frac{\sigma}{\rho} v_0 = 0 \quad \text{when } \rho = 1, \qquad \frac{v_0}{\rho} < \infty \quad \text{when } \rho = 0 \tag{1.4}$$

whereby only those solutions of (1.3,4) for which $v_0 \geqslant 0$ have physical meaning. (The membrane is subjected to tensile forces only.) Such solutions are called positive.

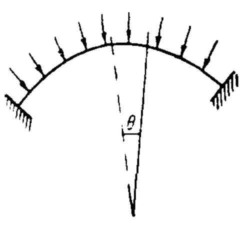

Fig. 1.

Theorems on the existence and uniqueness of positive solutions will be proved in Section 2. We remark that the solution of the problem (1.3,4) in the case of a spherical shell and uniform normal loading has been calculated approximately by Surkin [3]. It is natural to search for a solution of the problem (1.1,2) that is close to positive solutions of the problem (1.3,4). We will consider such solutions of problem (1.1,2) for which $v \geqslant 0$ and denote them membrane solutions. Moreover, it will be established that, for sufficiently small ε, such a solution exists and is unique. Indeed, as $\varepsilon \to 0$ the membrane solutions tend to positive solutions for the membrane.

At first sight it may seem paradoxical that, for example, in a spherical shell subjected to the action of normal external loading (Fig. 1), only tensile forces are produced. This can be explained by the fact that, since in this case the thin shell turns inside out (Fig. 2), the applied loading tends to increase the convexity of the shell.

For the proof of these facts firstly the formal asymptotic expansions of the solution of problem (1.1,2) are constructed, which are analogous to that obtained in the work [2] for the case of a plate (Section 3). In the vicinity of these expansions it is possible to apply Newton's method as extended to operator equations by Kantorovich [4]. Together with the derivation of the above-mentioned qualitative results, the asymptotic expansions constructed here also give a useful method of calculating membrane solutions.

Fig. 2.

We note that the case of the shell is essentially different from the case of the plate since the degenerate, as well as the non-degenerate problem has, in general, several solutions. A unique solution can be selected by means of the condition that the function v should be positive.

Below, for practical purposes, we will assume the following conditions

$$m_1\rho^2 \leqslant \varphi(\rho) \leqslant m_2\rho^2, \qquad \theta(\rho) \leqslant m_3\rho \qquad (m_1, m_2, m_3 = \text{const} > 0) \tag{1.5}$$

2. Membrane equations. We will prove that the problem (1.3,4) has just one positive solution. From (1.3) it follows that

$$u_0 = \theta - \frac{\varphi(\rho)}{v_0(\rho)} \tag{2.1}$$

The function $v_0(\rho)$ can be determined as the solution of the problem

$$Lv_0 \equiv -\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}\rho v_0 = \frac{\varphi^2}{2\rho v_0^2} - \frac{\theta^2}{2\rho} = 0$$

$$\frac{v_0}{\rho} < \infty \quad \text{when } \rho = 0, \qquad \frac{dv_0}{d\rho} - \frac{\sigma}{\rho} v_0 = 0 \quad \text{when } \rho = 1 \tag{2.2}$$

Theorem 2.1. Let the conditions (1.5) be satisfied. Then the problem (2.2) has no more than one positive solution.

In fact, if it is assumed that problem (2.2) has the two solutions $v_0(\rho) \geqslant 0$ and $v_0'(\rho) \geqslant 0$ then, with the notation $v_0 - v_0' = w$, we will have

$$\int_0^1 Lww\, d\rho \equiv \int_0^1 \left[\left(\frac{dw}{d\rho}\right)^2 + \frac{1}{2}\frac{w^2}{\rho^2}\right] d\rho + \left(\sigma - \frac{1}{2}\right) w^2(1) = -\int_0^1 \frac{(v_0 + v_0')\,\varphi^2 w^2}{2\rho v_0^2 v_0'^2}\, d\rho \tag{2.3}$$

By employing the simple inequality

$$w^2(1) = \left(\int_0^1 \frac{dw}{d\rho}\,d\rho\right)^2 \leqslant \int_0^1 \left(\frac{dw}{d\rho}\right)^2 d\rho \tag{2.4}$$

and by taking account of the positive definiteness of v_0 and v_0', we find from (2.3) that

$$\frac{1}{2}\int_0^1 \left[\left(\frac{dw}{d\rho}\right)^2 + \frac{w^2}{\rho^2}\right] d\rho + \sigma w^2(1) \leqslant 0 \tag{2.5}$$

Hence it follows that $w = v_0 - v_0' \equiv 0$. For the existence proof we make use of Chaplygin's method in a form extremely close to that which was developed by Babkin [5]; we thereby obtain at the same time an effective method of constructing the positive solutions.

Theorem 2.2. Problem (2.2) has no less than one positive solution.

Proof. Firstly, we observe that problem (2.2) is equivalent to the operator equation

$$v_0(\rho) = L^{-1}\left(\frac{\varphi^2}{2\rho v_0^2} - \frac{\theta^2}{2\rho}\right) \tag{2.6}$$

where

$$L^{-1}f \equiv \frac{1}{\rho}\int_0^\rho \eta \int_\eta^1 f(\xi)\,d\xi\,d\eta + \rho\frac{1+\sigma}{1-\sigma}\int_0^1 \eta \int_\eta^1 f(\xi)\,d\xi\,d\eta \tag{2.7}$$

We introduce the function $C(\rho)$ by the equality

$$C(\rho) = \left[\frac{\varphi^2(\rho)}{\theta^2 + \rho^2 a}\right]^{\frac{1}{2}} \tag{2.8}$$

where $a > 0$ is an arbitrary constant which satisfies the inequality

$$a^2\left(\max\frac{\theta^2(\rho)}{\rho^2} + a\right) \leqslant \left[\frac{16(1-\sigma)}{3-\sigma}\right]^2 \min\frac{\varphi^2}{\rho^4} \qquad (0 \leqslant \rho \leqslant 1) \tag{2.9}$$

A direct calculation shows that $C(\rho)$ satisfies the inequality

$$L^{-1}\left(\frac{\theta^2}{2\rho}\right) < L^{-1}\left(\frac{\varphi^2}{2\rho C^2}\right) \leqslant L^{-1}\left(\frac{\theta^2}{2\rho}\right) + C(\rho) \tag{2.10}$$

We will show that the solution of the problem is the limit of the sequence of functions $\{v_n\}$ determined by the relations

$$v_1 = L^{-1}\left(\frac{\varphi^2}{2\rho C^2} - \frac{\theta^2}{2\rho}\right), \quad v_{n+1} = v_n - \delta_n \quad (n = 1, 2, \ldots) \tag{2.11}$$

where δ_n is the solution of the equations

$$L\delta_n + M\delta_n - \alpha_n = 0, \qquad \frac{\delta_n}{\rho}\Big|_{\rho=0} < \infty, \qquad \frac{d\delta_n}{d\rho} - \frac{\sigma}{\rho}\delta_n\Big|_{\rho=1} = 0 \quad (2.12)$$

$$\alpha_n = Lv_n - \frac{\varphi^2}{2v_n^2\rho} + \frac{\theta^2}{2\rho}, \qquad M = \max\left|\frac{\varphi^2}{\rho v_1^3}\right| \qquad (0 \leqslant \rho \leqslant 1) \quad (2.13)$$

The quantity M is finite since, because of the condition $|\varphi| < m_1\rho^2$ and from (2.7,10,11), it follows that $v_1(\rho) \geqslant m_2\rho$.

We now verify that $\alpha_1 \leqslant 0$. In fact, by employing (2.11) and (2.10), we find

$$\alpha_1 = Lv_1 - \frac{\varphi^2}{2v_1^2\rho} + \frac{\theta^2}{2\rho} = \frac{\varphi^2(v_1^2 - C^2)}{2\rho v_1^2 C^2} \leqslant 0 \qquad (2.14)$$

By using this fact, we will show that $\delta_1 \leqslant 0$. Multiplying (2.12), for $n = 1$, by δ_1 and integrating with respect to ρ, we find

$$\int_0^1 \left[\left(\frac{d\delta_1}{d\rho}\right)^2 + \frac{1}{2}\frac{\delta_1^2}{\rho^2} + M\delta_1^2\right] d\rho + \left(\sigma - \frac{1}{2}\right)\delta_1^2(1) = \int_0^1 \alpha_1\delta_1\, d\rho \quad (2.15)$$

Estimating the left-hand side of (2.15) with the aid of the inequality (2.4), applied to δ_1 we are lead to

$$\int_0^1 \alpha_1\delta_1\, d\rho \geqslant 0 \qquad (2.16)$$

If now it is assumed that $\delta_1(\rho)$ is non-negative, it can be shown that, in any interval $[\xi_1, \xi_2] \subset [0, 1]$, $\delta_1(\rho) \geqslant 0$ for $\rho \in [\xi_1, \xi_2]$ and $\delta_1(\xi_1) = \delta_1(\xi_2) = 0$. But this leads to a controdiction, since, analogously to (2.16) for $[\xi_1, \xi_2]$, we obtain

$$\int_{\xi_1}^{\xi_2} \alpha_1\delta_1\, d\rho \geqslant 0 \qquad (2.17)$$

Thus, it has been proved that $\delta_1(\rho)$ is non-positive, i.e. that $\delta_1(\rho) \leqslant 0$.

From (2.15), by using (2.4) and the inequality

$$\int_0^1 \delta_1^2\, d\rho \leqslant \frac{1}{2}\int_0^1 \left(\frac{d\delta}{d\rho}\right)^2 d\rho$$

we are lead to

$$\left(M + \frac{3}{2} + 2\sigma\right)\|\delta_1\|_{L_2}^2 \leqslant \int_0^1 \alpha_1\delta_1\, d\rho \leqslant \|\alpha_1\|_{L_2}\|\delta_1\|_{L_2}$$

$$\|\delta_1\|_{L_2} \leqslant \frac{\|\alpha_1\|_{L_2}}{M_1}, \qquad M_1 = M + \frac{3}{2} + 2\sigma \tag{2.18}$$

We shall now show that $\alpha_2 \leqslant 0$. We have

$$\alpha_2 = Lv_2 - \frac{\varphi^2}{2v_2^2\rho} + \frac{\theta^2}{2\rho} = \frac{\varphi^2}{2\rho v_1^2} - \frac{\varphi^2}{2\rho\,(v_1 - \delta_1)^2} + M\delta_1 \tag{2.19}$$

By applying the Lagrange formula, we rewrite (2.19) in the form

$$\alpha_2 = \left[M - \frac{\varphi^2}{\rho\,(v_1 - \tau\delta_1)^3}\right]\delta_1 \qquad (0 \leqslant \tau \leqslant 1) \tag{2.20}$$

The fact that $\alpha_2(\rho)$ is non-positive follows from (2.20) by virtue of the definition (2.13) of the function M and the inequalities $v_1 \leqslant 0$, $\delta_1 \geqslant 0$. Moreover, (2.20) furnishes the following estimates

$$|\alpha_2(\rho)| \leqslant M\,|\delta_1(\rho)|, \qquad \|\alpha_2\|_{L_2} \leqslant M\,\|\delta_1\|_{L_2} \tag{2.21}$$

From (2.12) we obtain

$$\|\delta_2\|_{L_2} \leqslant \frac{\|\alpha_2\|_{L_2}}{M_1} \tag{2.22}$$

Similarly, it is possible to deduce the estimates

$$\|\delta_k\|_{L_2} \leqslant \frac{\|\alpha_k\|_{L_2}}{M_1}, \qquad \|\alpha_k\|_{L_2} \leqslant M\,\|\delta_{k-1}\|_{L_2} \qquad (k = 1, 2, \ldots) \tag{2.23}$$

Hence we obtain for arbitrary $k \geqslant 1$

$$\|\alpha_k\|_{L_2} \leqslant q^k \|\alpha_1\|_{L_2}, \quad \|\delta_k\|_{L_2} \leqslant q^k \frac{1}{M_1}\|\alpha_1\|_{L_2}, \quad q = \frac{M}{M_1} = \frac{M}{M + {}^3/_2 + 2\sigma} < 1 \tag{2.24}$$

We will prove that the series $v_1 - (\delta_1 + \delta_2 + \ldots)$, and hence the sequence $\{v_k\}$, converges, as well as the first derivative, uniformly on [0, 1] to some function v_0. For this we pass over from (2.12) to the equation

$$\delta_k = L^{-1}\alpha_k - ML^{-1}\delta_k \tag{2.25}$$

Now, from (2.24, 25), making use of (2.7), we obtain the estimate

$$\max_{0 \leqslant \rho \leqslant 1}\left(|\delta_k| + \left|\frac{d\delta_k}{d\rho}\right|\right) \leqslant m_3\left(\|\alpha_k\|_{L_2} + \|\delta_k\|_{L_2}\right) \leqslant m_4 q^k \|\alpha_1\|_{L_2} \qquad (k = 1, 2, \ldots) \tag{2.26}$$

Thus, the convergence has been established. It remains to be shown that v_0 is the solution of (2.6). From (2.13) results the following relation

$$v_k = L^{-1}\left(\frac{\varphi^2}{2\rho v_k^2} - \frac{\theta^2}{2\rho}\right) + L^{-1}(\alpha_k) \tag{2.27}$$

The last term in (2.27) converges uniformly to zero by virtue of (2.24) and the boundedness of L^{-1} as an operator acting from L_2 into the space of continuous functions C. Moreover, we note that $m\rho \geqslant v_k \geqslant v_1 \geqslant m_1\rho$, where m and m_1 are known constants. Therefore

$$\left|\frac{\varphi^2}{\rho v_k^2} - \frac{\varphi^2}{\rho v_0^2}\right| \leqslant m_5 \frac{\varphi^2}{\rho^5} |v_k - v_0| \leqslant m_6 \left|\frac{dv_k}{d\rho} - \frac{dv_0}{d\rho}\right| \to 0 \quad \text{when } k \to \infty$$

and the Equation (2.6) for v_0 can be obtained from (2.27) by the limit process $k \to \infty$.

We observe that it is possible to construct another proof of Theorem 2.2, by taking into account that problem (2.2) is equivalent to the problem of the minimum of the functional

$$I[v] = \frac{1}{2}\int_0^1 \left[\rho\left(\frac{dv}{d\rho}\right)^2 + \frac{v^2}{\rho} - 2\sigma v \frac{dv}{d\rho} + \frac{\varphi^2}{v} + \theta^2 v\right] d\rho \tag{2.28}$$

(the energy of the membrane) on the manifold of positive functions v which satisfy the boundary conditions (2.2). At the same time the Ritz method for calculating an approximate solution of problem (2.2) can be justified.

3. Construction of the asymptotic representation. We introduce the notation: Let the vector $\mathbf{V} \equiv (v, u)$ be the solution, and $P[\mathbf{V}]$ be the left-hand side the the system (1.1). For the solution (1.1,2) we will construct an asymptotic representation in the form

$$\begin{aligned} v &= \sum_{s=0}^{n+1} \varepsilon^s v_s + \sum_{s=0}^{n+1} \varepsilon^s h_s + \sum_{s=0}^{n+1} \varepsilon^s \alpha_s + x_n \\ u &= \sum_{s=0}^{n} \varepsilon^s u_s + \sum_{s=0}^{n} \varepsilon^s g_s + \sum_{s=0}^{n} \varepsilon^s \beta_s + z_n \end{aligned} \tag{3.1}$$

The functions $u_s(\rho)$, $v_s(\rho)$ will be obtained by means of the first iterative process [6].

Indeed, we assume that

$$\mathbf{V}_n \equiv (v^n, u^n) \qquad \left(v^n = \sum_{s=0}^{n} \varepsilon^s v_s, \quad u^n = \sum_{s=0}^{n} \varepsilon^s u_s\right) \tag{3.2}$$

and require that

$$P[\mathbf{V}_n] = O(\varepsilon^{n+1}) \tag{3.3}$$

By equating to zero the coefficients of ε^0, ε^1, ..., ε^n in (3.3), we obtain for the determination of v_0, u_0 the system of equations (1.3,4), and for the determination of v_s, u_s we obtain the system

$$Av_s - \frac{1}{2}\sum_{k+j=s} u_k u_j + \theta u_s = 0, \qquad \sum_{k+j=s} u_k v_j - \theta v_s + Au_{s-2} = 0$$
$$(s = 1, 2, \ldots, n+1;\ u_{-1} = 0) \tag{3.4}$$

with the boundary conditions

$$\left[\frac{v_s}{\rho}\right]_{\rho=0} < \infty, \qquad \left[\frac{dv_s}{d\rho} - \frac{\sigma}{\rho} v_s\right]_{\rho=1} = B_s$$

where B_s are as yet undetermined constants. The functions u_s, v_s do not satisfy the boundary conditions (1.2) for $\rho = 1$, and, consequently, the difference $\mathbf{V} - \mathbf{V}_n$ will not be small in the vicinity of the points $\rho = 1$. The resulting residuals in the fulfilment of the boundary conditions (1.2) when $\rho = 1$ can be compensated by functions of the boundary-layer type $h_s(\rho)$, $g_s(\rho)$, which can be determined with the aid of the second iteration process. Indeed, we will seek the difference $\mathbf{V} - \mathbf{V}_n$ in the form

$$v - v^n = \sum_{m=0}^{n} \varepsilon^m h_m, \qquad u - u^n = \sum_{m=0}^{n} \varepsilon^m g_m \tag{3.5}$$

Moreover, let

$$r = 1 - \rho, \qquad v_k = \sum_{l=0}^{\infty} v_{kl} r^l, \qquad u_k = \sum_{l=0}^{\infty} u_{kl} r^l, \qquad \theta = \sum_{l=0}^{\infty} \theta_l r^l$$

corresponding to the development in Taylor's series at the point $r = 0$. We substitute (3.5) into (1.1), make the substitution $\rho = 1 - \varepsilon t$, and equate the coefficients of ε^0, ε^1, ..., ε^n. With the calculation (3.3), this leads to the following system of linear differential equations with constant coefficients:

$$\frac{d^2 h_i}{dt^2} = 0 \qquad (i = 0, 1) \tag{3.6}$$

$$\frac{d^2 h_{s+2}}{dt^2} = R_1 h_{s+1} + R_2 h_s - \sum_{k+j+l=s} t^l u_{kl} g_j + \sum_{k+j+l+1=s} t^{l+1} u_{kl} g_j -$$
$$- \frac{1}{2}\sum_{i+j=s} g_i g_j + \frac{1}{2}\sum_{i+j+1=s} \mathrm{tan}_i g_j + \sum_{k+l=s} t^l \theta_l g_k - \sum_{k+l+1=s} t^{l+1} \theta_l g_k$$

$$\frac{d^2 g_s}{dt^2} - v_{00} g_s = R_1 g_{s-1} + R_2 g_{s-2} + \sum_{\substack{k+j+l=s \\ (s \neq j)}} t^l v_{kl} g_j - \sum_{k+j+l+1=s} t^{l+1} v_{kl} g_j +$$

$$+\sum_{j+m=s} g_j h_m - \sum_{j+m+1=s} \tan_j h_m + \sum_{k+m+l=s} t^l u_{kl} h_m - \sum_{k+m+l+1=s} t^{l+1} u_{kl} h_m -$$

$$-\sum_{k+l=s} t^l \theta_l h_k + \sum_{k+l-1=s} t^{l+1} \theta_l h_k \qquad (3.7)$$

where

$$R_1(\ ) \equiv 2t\frac{d^2(\ )}{dt^2} + \frac{d(\ )}{dt}, \qquad R_2(\ ) \equiv -t^2\frac{d^2(\ )}{dt^2} - t\frac{d(\ )}{dt} + (\ )$$

$$g_{-2} = g_{-1} = 0, \qquad v_{00} = \frac{1}{1-\sigma}\int_0^1 \eta \int_\eta^1 \left(\frac{\varphi^2}{2\xi v_0^2} - \frac{\theta^2}{2\xi}\right) d\xi\, d\eta > 0 \qquad (s = 0, 1, 2, \ldots)$$

Requiring that g_s makes up for the residue due to u_s in the fulfilment of the boundary conditions $u = 0$ when $\rho = 1$, we obtain the boundary conditions

$$g_s|_{t=0} = -u_{s0} \qquad (s = 0, 1, \ldots, n) \qquad (3.8)$$

The second boundary condition for g_s and the condition for h_s are obtained from the requirement that the solution has a boundary-layer character in the vicinity of $\rho = 1$

$$g_s|_{t=\infty} = 0, \qquad h_s|_{t=\infty} = 0 \qquad (s = 0, 1, \ldots, n) \qquad (3.9)$$

We determine the constants B_s by equating to zero the coefficients of $\varepsilon^s (s = 0, 1, \ldots, n + 1)$ in the equality

$$\sum_{s=0}^{n+1} \varepsilon^s \left[\frac{d(v_s + h_s)}{d\rho} - \frac{\sigma}{\rho}(v_s + h_s)\right]_{\rho=1} = 0 \qquad (3.10)$$

In particular, $B_0 = 0$. From (3.6,9), it is clear that $h_0 = h_1 = 0$. Indeed, hence follows the correctness of the choice of the boundary conditions (1.4) for the positive solution in the problem of the equilibrium of the membrane (1.3,4). From (3.7) to (3.9) we obtain, by virtue of (3.6), when $s = 0$

$$\frac{d^2 g_0}{dt^2} - v_{00} g_0 = 0, \quad g_0|_{t=0} = -u_{00}, \quad g_0|_{t=\infty} = 0, \quad v_{00} > 0$$

$$g_0 = -u_{00}\exp(-\sqrt{v_{00}}\,t) = -u_0(1)\exp\left[-\sqrt{v_0(1)}\,\frac{1-\rho}{\varepsilon}\right] \qquad (3.11)$$

i.e. g_0 has the nature of a boundary-layer function of zero order. Now we determine h_2. From (3.7), (3.9) and (3.11) we obtain

$$\frac{d^2h_2}{dt_2} + u_0(1)\,g_0 + \frac{1}{2}g_0^2 - \theta(1)\,g_0 = 0, \qquad h_2|_{t=\infty} = 0$$

$$h_2 = -\frac{[u_0(1) - \theta(1)]^2}{v_0(1)}\left(\frac{1}{8}\exp\left[-2\sqrt{v_0(1)}\,\frac{1-\rho}{\varepsilon}\right] - \exp\left[-\sqrt{v_0(1)}\,\frac{1-\rho}{\varepsilon}\right]\right)$$

Moreover, from the condition (3.10), equating to zero the coefficient of ε^1, we find

$$B_1 = -\frac{3}{4}\,\frac{[u_0(1) - \theta(1)]^2}{\sqrt{v_0(1)}}$$

The functions g_s can be determined from equations which are of the same type as (3.11) but nonhomogeneous.

The infinitely differentiable, non-increasing functions $\alpha_s(\rho)$ and $\beta_s(\rho)$ compensate for the disparities in the satisfaction of the boundary conditions when $\rho = 0$, which are associated with the functions $g_s(\rho)$ and $h_s(\rho)$, and are

$$\alpha_s(\rho) = \begin{cases} -h_s(0) & (0 \leqslant \rho \leqslant 0.1), \\ 0 & (0.2 \leqslant \rho \leqslant 1), \end{cases} \qquad \beta_s(\rho) = \begin{cases} -g_s(0) & (0 \leqslant \rho \leqslant 0.1) \\ 0 & (0.2 \leqslant \rho \leqslant 1) \end{cases}$$

Thus the process of constructing an asymptotic representation reduces to the following. We find the positive solution v_0, u_0 of the problem (1.3,4), and from (3.11) we determine g_0. Then, from (3.4) we successively find v_s, u_s, and from (3.7) to (3.9) we find h_s, g_s ($s = 1, 2, \ldots$).

4. Justification of the asymptotic expansions. Existence of the membrane solution. We introduce the notation $\varphi_k = v - x_k$, $\psi_k = u - z_k$.

Lemma 4.1. For φ_k and ψ_k we have the valid estimates

$$A\varphi_k - \frac{1}{2}\psi_k^2 + \theta\psi_k = O(\rho\varepsilon^{k+1})\,^*, \quad \varepsilon^2 A\psi_k + \varphi_k\psi_k - \theta\psi_k + \varphi(\rho) = O(\rho\varepsilon^{k+1}) \tag{4.1}$$

(the condition $f(\rho, \varepsilon) = O(\rho\varepsilon^{k+1})$ means that $|f(\rho, \varepsilon)| \leqslant m\rho\varepsilon^{k+1}$).

We omit the proof, since it is almost a literal repetition of the proof in the case of a plate.

Lemma 4.2. For sufficiently small $\varepsilon (0 < \varepsilon < \varepsilon_1)$ for all $\rho \in [0, 1]$ the following inequalities are valid

$$\varphi_k \geqslant 0, \qquad \frac{\varphi_k}{\rho} > \frac{v_1}{2\rho} \geqslant m > 0 \tag{4.2}$$

Here v_1 is defined by (2.11). We have

$$\varphi_k = \sum_{s=0}^{k+1} \varepsilon^s v_s + \sum_{s=0}^{k+1} \varepsilon^s h_s^\circ, \qquad h_s^\circ = h_s + \alpha_s$$

Taking into account that $h_0 = 0$, as well as the estimates $v_s(\rho) = O(\rho)$, $h_s^\circ(\rho) = O(\rho)$, we have

$$\varphi_k = v_0 + O(\rho\varepsilon) \tag{4.3}$$

Now the inequality (4.2) follows immediately from (4.3), if use is made of the relation $v_0 \geqslant v_1 \geqslant m\rho$ which was mentioned during the proof of Theorem 2.2.

We introduce the Banach space of the vectors $\mathbf{V} \equiv (v, u)$:

1) consiting of the vectors with the finite norm

$$(L_\rho) \qquad \|\mathbf{V}\|_{L_\rho}^2 = \int_0^1 \frac{v^2 + u^2}{\rho} d\rho \tag{4.4}$$

2) the closure of the manifold of smooth vector-functions, satisfying the conditions (1.2), by the norm

$$(W_\rho) \qquad \|\mathbf{V}\|_{W_\rho}^2 = \int_0^1 \frac{1}{\rho} [(Av)^2 + (Au)^2] d\rho \tag{4.5}$$

The problem (1.1,2) will be treated as the functional equation

$$P[\mathbf{V}] = 0 \tag{4.6}$$

where the operator P is defined by the left-hand side of the system (1.1) and operates from W_ρ into L_ρ.

Theorem 4.1. The problem (1.1,2) has one and only one membrane solution. The uniqueness is proved in exactly the same way as in the case of the plate [7]. For the proof of existence use is made of a theorem of Kantorovich [4] on the convergence of Newton's method. For the initial approximation, $\mathbf{V}_k^* = (\varphi_k, \psi_k)$ is taken.

In the application to the present problem, the theorem is formulated in the following manner.

Theorem 4.2. Suppose the operator P has been defined inside the sphere $\Omega(\|\mathbf{V} - \mathbf{V}_k^*\| \leqslant R)$ of the space W_ρ and that in the closed sphere $\Omega_0(\|\mathbf{V} - \mathbf{V}_k^*\| \leqslant r)$ it has a continuous second derivative. Suppose also that:

1) there exists a linear operation $\Gamma_0 = [P_{\mathbf{V}_k^*}'(\mathbf{V})]^{-1}$

2) $\| \Gamma_0(P\,[V_k^*]) \|_{W_\rho} \leqslant \eta$

3) $\| \Gamma_0 P''(V) \|_{W_\rho} \leqslant K \qquad (V \in \Omega_0)$

4) $h = K\eta \leqslant \frac{1}{2}, \qquad r \geqslant r_0 = \frac{1 - \sqrt{1 - 2h}}{h}\eta$

Then Equation (4.6) has the solution V^*, to which Newton's process converges. In this

$$\| V^* - V_k^* \|_{W_\rho} \leqslant r_0 \tag{4.7}$$

It is clear that the conditions of the theorem are satisfied if

$$\| P(V_k^*) \|_{L_\rho} \| [P_{V_k^*}']^{-1} \|^2 \| P_V'' \| \leqslant \frac{1}{2} \tag{4.8}$$

We will now prove that (4.8) is satisfied for sufficiently small ε and for arbitrary $k > 3$. From (4.1) we deduce

$$\| P(V_k^*) \|_{L_\rho} \leqslant m \varepsilon^{k+1} \tag{4.9}$$

We make an estimate of the second factor in (4.8). We have

$$P_{V_k^*}'(V) \equiv (Av - \psi_k u + \theta u,\ \varepsilon^2 Au + \psi_k v + \varphi_k u - \theta v) \tag{4.10}$$

From (4.10) we obtain

$$\int_0^1 \frac{1}{\rho} P_{V_k^*}'(V)\, V\, d\rho = \int_0^1 \left[\left(\frac{dv}{d\rho}\right)^2 + \frac{1}{2}\frac{v^2}{\rho^2}\right] d\rho + \varepsilon^2 \int_0^1 \left[\left(\frac{du}{d\rho}\right)^2 + \frac{1}{2}\frac{u^2}{\rho^2}\right] d\rho + + \left(\sigma - \frac{1}{2}\right) v^2(1) + \int_0^1 \frac{\varphi_k u^2}{\rho}\, d\rho \tag{4.11}$$

From (4.11), by using (4.2) and (2.4), we deduce

$$\int_0^1 \frac{1}{\rho} P_{V_k^*}'(V)\, V\, d\rho \geqslant \varepsilon^2 \| V \|_{L_\rho}^2$$

Then, it follows that

$$\| P_{V_k^*}'(V) \|_{L_\rho} \geqslant \varepsilon^2 \| V \|_{L_\rho} \tag{4.12}$$

Applying the inequality (4.12), it is not difficult to prove that the operator $P_{V_k^*}'$ has an inverse and that we have the estimate

$$\| [P_{V_k^*}']^{-1} \| \leqslant \frac{1}{\varepsilon^2} \tag{4.13}$$

For the estimate of $\| P_V'' \|$, we consider the bi-linear form

$$P_{\mathbf{V}}''(\mathbf{V}')(\mathbf{V}'') = (-u'u'', u'v'' + u''v') \tag{4.14}$$

Now we make note of the validity of inequalities of the form of "imbedding theorems"*

$$\int_0^1 \frac{v^4 + u^4}{\rho} d\rho \leqslant m \|\mathbf{V}\|_{W_\rho}^4, \qquad \max_{0 \leqslant \rho \leqslant 1} |v| \leqslant m \|\mathbf{V}\|_{W_\rho} \tag{4.15}$$

which are easily deduced from (4.5) and the integral representations of u, v in terms of Au and Av respectively. Therefore

$$\|P_{\mathbf{V}}''(\mathbf{V}')(\mathbf{V}'')\|_{L_\rho}^2 \leqslant m^2 \|\mathbf{V}'\|_{W_\rho}^2 \|\mathbf{V}''\|_{W_\rho}^2 \tag{4.16}$$

Whence there results the estimate

$$\|P_{\mathbf{V}}''\| \leqslant m_1 \tag{4.17}$$

From (4.9), (4.13) and (4.17) we obtain

$$\|P[\mathbf{V}_k^*]\|_{L_\rho} \|[O'_{\mathbf{V}_k^*}]^{-1}\|^2 \|P_{\mathbf{V}}''\| \leqslant m_2 \varepsilon^{k-3} < \frac{1}{2} \tag{4.18}$$

if $k > 3$ and ε is sufficiently small ($0 < \varepsilon < \varepsilon_1$). Hence, the conditions of the theorem of Kantorovich are satisfied. Therefore, Equation (4.6), which is equivalent to the problem (1.1,2), has the solution $\mathbf{V}^* = (v, u)$ for which an estimate of the form (4.7) holds

$$\|\mathbf{V}^* - \mathbf{V}_k^*\|_{W_\rho} \leqslant r_0$$

Calculating the value of r_0 with the aid of the inequalities (4.9) and (4.13), we find

$$\|\mathbf{V}^* - \mathbf{V}_k^*\|_{W_\rho} \leqslant m_3 \varepsilon^{k-1} \qquad (k > 3) \tag{4.19}$$

Finally, from (4.19), with the aid of (4.3) and (4.15), we obtain

$$v = v_0 + O(\rho\varepsilon) \tag{4.20}$$

Hence it follows that $v \geqslant m\rho$, if ε is sufficiently small. This means that the constructed solution $\mathbf{V}^*$ is a membrane one. Theorem 4.1 has been proved.

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* *Translator's note:* In the Russian literature this expression refers to a number of theorems attributed to S.L. Sobolev. See, for instance, Smirnov; *Kurs Vysshei Matematiki*, Vol. 5, Section 114.

The condition $v \geqslant 0$ allows the application of the reasoning in the work [2], and the following conclusions are obtained.

Theorem 4.3. For the membrane solution of problem (1.1,2) the asymptotic representations (3.1) are valid, and, moreover, the remainders are bounded by the following estimates:

$$\max_{0\leqslant\rho\leqslant 1} |x_k(\rho)| \leqslant m_1\varepsilon^{k+1}, \qquad \max_{0\leqslant\rho\leqslant 1} |z_k(\rho)| \leqslant m_2\varepsilon^{k+\frac{1}{2}} \qquad (k=0,1,\ldots)$$

$$\max_{0\leqslant\rho\leqslant 1} \left|\frac{dx_k}{d\rho}\right| \leqslant m_3\varepsilon^{k+1} \quad (k=0,1,\ldots), \qquad \max_{0\leqslant\rho\leqslant 1} \left|\frac{dz_k}{d\rho}\right| \leqslant m_4\varepsilon^{k-1} \quad (k=2,3,\ldots)$$

$$\max_{0\leqslant\rho\leqslant 1} \left|\frac{d^2x_k}{d\rho^2}\right| \leqslant m_5\varepsilon^{k-\frac{1}{2}} \quad (k=1,2,\ldots), \quad \max_{0\leqslant\rho\leqslant 1} \left|\frac{d^2z_k}{d\rho^2}\right| \leqslant m_6\varepsilon^{k-\frac{5}{2}} \quad (k=3,4,\ldots)$$

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